

Quantized Dirac Operators

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We determine what should correspond to the Dirac operator on certain quantized hermitian symmetric spaces and what its properties are. A new insight into the quantized wave operator is obtained.

1 Introduction

In the articles [3], [1], [2], [4], [5], and [6] it was shown how intertwining differential operators on spaces of vector valued holomorphic functions on hermitian symmetric spaces of the non-compact type are intimately connected both with homomorphisms between generalized Verma modules and with singular unitary representations through what was called the “missing k types”. It was clear from the outset that the construction was applicable in a much broader context for classical (semi-simple) Lie algebras. Later, it was natural to extend it further to quantum groups, in particular after the classification of unitarizable highest weight modules was obtained [7].

In connection with the generalization to quantum groups of the above, one needs of course to find good replacements of hermitian symmetric spaces, holomorphic functions, and, finally, differential operators. We believe that we through the discovery of quadratic algebras associated to hermitian symmetric spaces, see e.g. [8], have found the right setting.

In a series of papers, c.f. [10], [11], and references cited therein, Dobrev has studied a quantization of the d’Alembert operator and its invariance properties and has also addressed the issue of relevant spaces. The quantized wave operator is also discussed in [9] which has some overlap with Dobrev’s work, both mathematical and notational, but in fact, Dobrev’s methods have much more in common with the earlier work cited above. Even more so, the methods in those references reach much further as we will illustrate below by giving “the quantization of the Dirac operator” in the setting of quantum groups. Our approach shows that the quantized wave operator is a major part of the quantized Dirac operator; in some sense the term that represents the most “non-commutative” part.

2 Basics

For any hermitian symmetric space there is a (matrix valued) first order holomorphic differential operator that intertwines two unitary highest weight representations. Sometimes there are even two. These operators might be called “Dirac operators” and they all have quantizations. In order to keep the presentation simple

and at the same time treat the most relevant case for physics, we will only give the details here in the case of $su(2, 2)$ and its quantized enveloping algebra $\mathcal{U}_q(su(2, 2))$. To be specific, $\mathcal{U}_q = \mathcal{U}_q(su(2, 2))$ is generated by

$$E_\mu, E_\nu, E_\beta, F_\mu, F_\nu, F_\beta, K_\nu^{\pm 1}, K_\mu^{\pm 1}, K_\beta^{\pm 1}, \quad (1)$$

with the usual relations of which we only list those we need in the following:

$$\begin{aligned} F_\nu^2 F_\beta + (q + q^{-1}) F_\nu F_\beta F_\nu + F_\beta F_\nu^2 &= 0, & F_\mu^2 F_\beta + (q + q^{-1}) F_\mu F_\beta F_\mu + F_\beta F_\mu^2 &= 0, \\ F_\beta^2 F_\nu + (q + q^{-1}) F_\beta F_\nu F_\beta + F_\nu F_\beta^2 &= 0, & F_\beta^2 F_\mu + (q + q^{-1}) F_\beta F_\mu F_\beta + F_\mu F_\beta^2 &= 0, \\ F_\nu F_\mu &= F_\mu F_\nu, & K_\nu F_\beta &= q F_\beta K_\nu, & K_\mu F_\beta &= q F_\beta K_\mu, & K_\beta F_\beta &= q^{-2} F_\beta K_\beta, \\ [E_\beta, F_\beta] &= \frac{K_\beta - K_\beta^{-1}}{q - q^{-1}}, & [E_\mu, F_\mu] &= \frac{K_\mu - K_\mu^{-1}}{q - q^{-1}}. \end{aligned} \quad (2)$$

$$\begin{aligned} \triangle(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \triangle(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, \\ \triangle(K_i) &= K_i \otimes K_i, & S(E_i) &= -K_i^{-1} E_i, & S(F_i) &= -F_i K_i, & S(K_i) &= K_i^{-1}, \text{ and} \\ \epsilon(E_i) &= \epsilon(F_i) = 0, & \epsilon(K_i) &= 1. \end{aligned} \quad (3)$$

2.1 The quadratic algebra

Let $w_1 = F_\beta, w_2 = F_\mu F_\beta - q F_\beta F_\mu, w_3 = F_\nu F_\beta - q F_\beta F_\nu, w_4 = F_\mu w_3 - q w_3 F_\mu$. Then the four w_i s generate a quadratic algebra denoted \mathcal{A}_q . The relations are the usual ones,

$$\begin{aligned} w_1 w_2 &= q w_2 w_1, & w_1 w_3 &= q w_3 w_1, & w_3 w_4 &= q w_4 w_3, \\ w_2 w_4 &= q w_4 w_2, & w_2 w_3 &= w_3 w_2, & w_1 w_4 - w_4 w_1 &= (q - q^{-1}) w_2 w_3. \end{aligned} \quad (4)$$

The element $w_1 w_4 - q w_2 w_3$ is in the center of \mathcal{A}_q , and at a generic q , the center is generated by this element.

There is an action (this is the co-adjoint action $k \star w = \sum_i b_i w S(a_i)$ if $\triangle k = \sum_i a_i \otimes b_i$) of the algebra $\mathcal{U}_q(su(2) \times su(2))$ on \mathcal{A}_q given by

$$\begin{aligned} F_\mu \star w_1 &= w_2, & F_\mu \star w_3 &= w_4, & F_\mu \star w_2 &= 0, & F_\mu \star w_4 &= 0, \\ E_\mu \star w_2 &= w_1, & E_\mu \star w_4 &= w_3, & E_\mu \star w_1 &= 0, & E_\mu \star w_3 &= 0, \\ K_\mu \star w_1 &= q w_1, & K_\mu \star w_3 &= q w_3, & K_\mu \star w_2 &= q^{-1} w_2, & K_\mu \star w_4 &= q^{-1} w_4. \end{aligned} \quad (5)$$

This action is derived from a left action in the quantized enveloping algebra but is not identical with this action. For later use we record here the identities $F_\mu w_2 = q^{-1} w_2 F_\mu$ and $E_\mu w_2 = w_1 K_\mu^{-1}$.

There are analogous equations for the index ν obtainable simply by the interchange $2 \leftrightarrow 3$.

Observe that

$$w_4^N w_1 = w_1 w_4^N - (q - q^{-2N+1}) w_2 w_3 w_4^{N-1}. \quad (6)$$

Later on, we shall use functions defined on the quadratic algebra \mathcal{A}_q , in particular polynomial functions. Actually, we wish to work with functions on \mathbf{C}^4 . For this reason we introduce the symbols

$$[\gamma]_q! = [\gamma_1]_q! [\gamma_2]_q! [\gamma_3]_q! [\gamma_4]_q!, \quad (7)$$

where for any non-negative integer n , $[n]_q!$ denotes the usual q factorial.

Likewise, $w^\gamma = w_1^{\gamma_1} w_2^{\gamma_2} w_3^{\gamma_3} w_4^{\gamma_4}$ in \mathcal{A}_q , and if $z = (z_1, z_2, z_3, z_4) \in \mathbf{C}^4$ then $z^\gamma = z_1^{\gamma_1} z_2^{\gamma_2} z_3^{\gamma_3} z_4^{\gamma_4}$.

To a (polynomial) function f on \mathcal{A}_q we now introduce the function Ψ_f on \mathbf{C}^4 by

$$\Psi_f(z) = f \left(\sum_{\gamma} \frac{z^\gamma}{[\gamma]_q} w^\gamma \right). \quad (8)$$

The motivation for this normalization is perhaps a little flimsy, but we mention that the so-called “divided powers” algebra is more fundamental than others. Furthermore, the space of functions thus defined on \mathbf{C}^4 actually comes equipped with an associative \star -product. See [9] for further details.

Later on we shall encounter right multiplication operators on the space of functions on \mathcal{A}_q , i.e. operations of the form $((w_0^R)^\dagger \cdot f)(w^\gamma) = f(w^\gamma w_0)$. Below, w_0 will be either w_1, w_2, w_3, w_4 , or $w_1 w_4 - q w_2 w_3$. Transformed into operators on functions on \mathbf{C}^4 via (8) these become q -differential operators expressible in terms of, among other things, the usual q -differential operators on \mathbf{C}^1 as well as scaling operators (c.f. [9], [10]),

$$\left(\left[\frac{\partial}{\partial x} \right]_q \psi \right)(x) = \frac{f(xq) - f(xq^{-1})}{q - q^{-1}}, \text{ and} \quad (9)$$

$$K_i(z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) = q^{-\alpha_i} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} \quad i = 1, 2, 3, 4. \quad (10)$$

In terms of un-quantized operators, if $q = e^{\hbar}$ then $K_i = \sum_n (-\hbar S_i)^n$, where $S_i = z_i \frac{\partial}{\partial z_i}$. Also observe that $K_i = \left[\frac{\partial}{\partial z_i} \right]_q z_i - q z_i \left[\frac{\partial}{\partial z_i} \right]_q$.

We obtain:

$$(w_4^R)^\dagger = \left[\frac{\partial}{\partial z_4} \right]_q, \quad (w_2^R)^\dagger = K_4 \left[\frac{\partial}{\partial z_2} \right]_q, \quad (w_3^R)^\dagger = K_4 \left[\frac{\partial}{\partial z_3} \right]_q, \text{ and} \quad (11)$$

$$(w_1^R)^\dagger = K_2 K_3 K_4^2 \left[\frac{\partial}{\partial z_1} \right]_q + z_4 (1 - q^{-2}) K_4 \square_q, \text{ where} \quad (12)$$

$$\square_q \stackrel{Def.}{=} (w_1 w_4 - q w_2 w_3)^\dagger = K_2 K_3 \left[\frac{\partial}{\partial z_1} \right]_q \left[\frac{\partial}{\partial z_4} \right]_q - q \left[\frac{\partial}{\partial z_2} \right]_q \left[\frac{\partial}{\partial z_3} \right]_q. \quad (13)$$

The last defined operator is of course nothing else but the quantized wave operator. At the most singular point of unitarity for representations in spaces of scalar valued functions it is an intertwining differential operator.

3 Induced representations

Let \mathcal{B}^+ denote the part of \mathcal{U}_q generated by the elements $K_i^{\pm 1}, E_\beta, E_\nu, E_\mu$, and let \mathcal{B}^- be defined analogously. Let χ_λ be a 1-dimensional representation of \mathcal{B}^+ corresponding to the weight λ . Let

$$H^0(\chi_\lambda) = \{f : \mathcal{U}_q \mapsto \mathcal{C} \mid f(ub^+) = \chi_\lambda^{-1}(b^+)f(u) \text{ and } f \in \text{FIN}(\mathcal{U}_q(su(2) \times su(2)))\} \quad (14)$$

with the action $(x \cdot f)(u) = f(S(x)u)$, S being the antipode. The condition FIN denotes the usual finiteness condition.

Clearly, any element $u_0 \in \mathcal{U}_q$ may in some sense be viewed as an intertwiner through the map $f \mapsto f_{u_0}$ where

$$f_{u_0}(u) = f(u \cdot u_0), \quad (15)$$

but if we further want the target space to be of the form $H^0(\chi_{\lambda_1})$ for some other 1-dimensional representation of \mathcal{B}^+ , we can assume that $u_0 \in \mathcal{B}^-$ (in fact \mathcal{N}^-) and we must then clearly have that

$$\forall b^+, u : f(u \cdot b^+ \cdot u_0) = \chi_{\lambda_1}^{-1}(b^+)f(u \cdot u_0). \quad (16)$$

This, on the other hand can be directly translated into the following statement:

Let M_λ be the Verma module of highest weight λ and let $0 \neq v_\lambda$ denote the highest weight vector. Then $u_0 \cdot v_\lambda$ must define a primitive weight vector in M_λ .

Theorem 3.1 (H-J, [1], [2], [6]) *There is a bijective correspondence between intertwining differential operators and homomorphisms between generalized Verma modules.*

4 Dirac operators

In the present case we shall study two representations and intertwiners (which will be the quantized Dirac operators) thereon.

Suppose $\chi_\lambda^1(K_\mu) = q$, $\chi_\lambda^1(K_\nu) = 1$, $\chi_\lambda^1(K_\beta) = q^x$. Then the FIN condition implies that a function $f \in H^0(\chi_\lambda^1)$ is uniquely determined on elements of the form $p_1(w) + p_2(w)F_\mu$ in \mathcal{U}_q , the elements p_1, p_2 being polynomials in \mathcal{A}_q . Via the correspondence (8), $H^0(\chi_\lambda^1)$ is then identified with the space $\mathcal{P} \otimes \mathcal{C}^2$, \mathcal{P} being the usual space of polynomials on \mathcal{C}^4 . It is now easy to see that an element

$$u_0^+ = F_\mu^- F_\beta - (q + q^{-1}) \cdot F_\beta E_\mu = w_2 - q^{-1} \cdot w_1 F_\mu \quad (17)$$

is an intertwiner into a space $H^0(\chi_\delta)$ provided $q^{2x-4} = 1$. In this case, $\chi_\delta(K_\mu) = 1$, $\chi_\delta(K_\nu) = q$, and $\chi_\delta(K_\beta) = q^{x+1}$. In the case where $x = 2$ and q is real there is a unitary highest weight representation in a subspace (the kernel of the operator D^+ below). If $x \neq 2$ we may still have an intertwiner provided that $q = e^{\frac{2\pi i \cdot p}{2x-4}}$ for some integer p .

The target space thus consists of functions that are non-zero only on expressions of the form $p_1(w) + p_2(w)F_\nu$. Now compute, for $f \in H^0(\chi_\lambda^1)$,

$$\begin{aligned} f_{u_0^+}(p_1(w) + p_2(w)F_\nu) &= f([p_1(w) + p_2(w)F_\nu][w_2 - q^{-1}w_1F_\mu]) \\ &= f([p_1(w)w_2 + p_2(w)w_4] - q^{-1}[p_1(w)w_1 + p_2(w)w_3]F_\mu). \end{aligned} \quad (18)$$

Thus, the intertwiner $f \mapsto f_{u_0^+}$ may be expressed as a map $D^+ : \mathcal{P} \otimes \mathcal{C}^2 \mapsto \mathcal{P} \otimes \mathcal{C}^2$ given by the matrix

$$\begin{aligned} D^+ &= \begin{pmatrix} (w_2^R)^\dagger & (w_4^R)^\dagger \\ -q^{-1}(w_1^R)^\dagger & -q^{-1}(w_3^R)^\dagger \end{pmatrix} \\ &= \begin{pmatrix} K_4 \left[\frac{\partial}{\partial z_2} \right]_q & \left[\frac{\partial}{\partial z_4} \right]_q \\ -q^{-1}K_2K_3K_4^2 \left[\frac{\partial}{\partial z_1} \right]_q & -q^{-1}K_4 \left[\frac{\partial}{\partial z_3} \right]_q \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ -q^{-1}z_4(1 - q^{-2})K_4\Box_q & 0 \end{pmatrix}. \end{aligned} \quad (19)$$

Of course, there is an analogous operator D^- obtainable by interchanging μ and ν ;

$$\begin{aligned} D^- &= \begin{pmatrix} (w_3^R)^\dagger & (w_4^R)^\dagger \\ -q^{-1}(w_1^R)^\dagger & -q^{-1}(w_2^R)^\dagger \end{pmatrix} \\ &= \begin{pmatrix} K_4 \left[\frac{\partial}{\partial z_3} \right]_q & \left[\frac{\partial}{\partial z_4} \right]_q \\ -q^{-1}K_2K_3K_4^2 \left[\frac{\partial}{\partial z_1} \right]_q & -q^{-1}K_4 \left[\frac{\partial}{\partial z_2} \right]_q \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ -q^{-1}z_4(1 - q^{-2})K_4\Box_q & 0 \end{pmatrix}. \end{aligned} \quad (20)$$

It is easy to see that $D^+D^- = D^-D^+ = -q^{-1} \begin{pmatrix} \Box_q & 0 \\ 0 & \Box_q \end{pmatrix}$.

With the exception of the term $-q^{-1}z_4(1 - q^{-2})K_4\Box_q$, the operators D^\pm look like any simple-minded generalization of the Dirac operator to the q realm. However, there **is** the extra term which represents the departure from the “quasi-commutative” situation. We have in [9] seen the first indications of an auxiliary bundle, which in fact can be constructed using \Box_q , in which the new first order differential operators are the covariant derivatives. We leave this point to subsequent investigations.

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